Vol. 11(8), pp. 109-113, November 2018
DOI: 10.5897/AJMCSR2018.0764
Article Number: F5F6E2B59269
ISSN: 2006-9731

http://www.academicjournals.org/AJMCSR

ACADEMIC JOURNALS
expand your knowledge

African Journal of Mathematics and Computer Science Research

Full Length Research Paper

Proof of Beal's conjecture

Samuel Bonaya Buya

Ngao Girls' Secondary School, Kenya.

Received 6 October 2018; Accepted 7 November, 2018

In this research, a proof of Beal's conjecture is presented. A possible Pythagorean algebraic relationship between the terms of the conjecture problem will be proposed and used to arrive at the proof results. In the process of seeking the proof the solution of the congruent number problem through a family of cubic curves will be discussed.

Key words: Proof of Beal's conjecture, proof of ABC conjecture, algebraic proof of Fermat's last theorem, the congruent number problem, rational points on the elliptic curve, Pythagorean triples

INTRODUCTION

Beal's conjecture was formulated in 1993 by Andrew Beal, a banker and amateur mathematician while investigating generalizations of Fermat's last theorem. The conjecture was formulated after some extensive computational experiments were conducted in August 1993. In some publications the conjecture has been occasionally referred to as the generalized Fermat's last theorem (Mauldin, 1997), (Bennet et al, 2014) the Mauldin conjecture (puzzles, n.d.) and the Tijdeman-Zagier conjecture (puzzles, n.d.)], (Waldschmidt, 2004), (Wikipedia, 2018).

The conjecture states that:

If $A^x + B^y = C^z$, where A, B, C are positive integers and x, y and z are all positive integers greater 2, then A, B and C have a common prime factor.

To prove the conjecture a Pythagorean algebraic relationship between the terms of the conjecture will be derived and used.

An algebraic relationship between the terms of Pythagorean triples

Consider some Pythagorean integer triples x, y and z related by the equation:

$$x^2 + y^2 = z^2$$
(1)

If $y = \dfrac{\frac{x^2}{a} - a}{2}$ (2)

And again $z = \dfrac{\frac{x^2}{a} + a}{2}$ (3)

*Corresponding author. E-mail: sbonayab@gmail.com

Then by Equations 2 and 3

$$z-y=a \tag{4}$$

$$z+y=\frac{x^2}{a} \tag{5}$$

Multiplying Equations 5 and 4 together:

$$x^2=(z+y)(z-y)=z^2-y^2=\left(\frac{\frac{x^2}{a}+a}{2}\right)^2-\left(\frac{\frac{x^2}{a}-a}{2}\right)^2 \rightarrow$$

$$x^2+\left(\frac{\frac{x^2}{a}-a}{2}\right)^2=\left(\frac{\frac{x^2}{a}+a}{2}\right)^2 \tag{6}$$

A relationship has been established between the terms of a Pythagorean triple. Some forms of the above Pythagorean relationships were first discussed in a paper entitled simple algebraic proofs of Fermat's last theorem (Buya, 2017a)

Solution of the congruent number problem

The technique used in proving Beal conjecture can be used to solve problems in number theory, arithmetic of elliptic of elliptic curves etc. (Elkies, 2007). An important problem in number theory that can be solved is the congruent number theory (Buya, 2017b).
The results in section 2 can be used to solve the congruent number problem. If the perpendicular height of a right angled triangle is x then by equation 7 its base is $\frac{\frac{x^2}{a}-a}{2}$. The area of a congruent triangle is given by:

$$A=x\left(\frac{\frac{x^2}{a}-a}{4}\right)=\frac{x^3-a^2x}{4a} \tag{1}$$

For the purpose of solution of the congruent number problem (a) is considered to be a rational number in this section.
Thus the family of curves of the cubic equation

$$y=\frac{x^3-a^2x}{4a} \tag{2}$$

has an infinite number of points for solution of the congruent number problem. In the function 2 above the y coordinate represents the area of a congruent number with (**x**) and (**a**) as rational numbers.
Thus in the elliptic curve:

$$y^2=x^3-n^2x \tag{3},$$

where n is congruent number if n is of the form $n=\frac{b^3-a^2b}{4a}\wedge a,b\in\frac{p}{q}\wedge p,q\in N\wedge q\neq 0$

Consider the elliptic curve 3. Making the congruent number (n) the subject of the equation we obtain the following:

$$n^2=x^2-\frac{y^2}{x}=\left(x+\frac{y}{\sqrt{x}}\right)\left(x-\frac{y}{\sqrt{x}}\right) \tag{4}$$

For n to be a congruent number take:

$$x+\frac{y}{\sqrt{x}}=a^2 \tag{5}$$

$$x-\frac{y}{\sqrt{x}}=\frac{1}{4}\left(\frac{a-b^2}{2b}\right)^2 \tag{6}$$

From Equations 5 and 6

$$x=\frac{1}{2}\left(a^2+\frac{1}{4}\left(\frac{a-b^2}{2b}\right)^2\right) \tag{7}$$

$$y=\frac{1}{4}\left(a^2-\frac{1}{4}\left(\frac{a-b^2}{2b}\right)^2\right)\left(\sqrt{\frac{1}{2}\left(a^2+\frac{1}{4}\left(\frac{a-b^2}{2b}\right)^2\right)}\right) \tag{8}$$

For a rational number on the elliptic curve 3 select rational numbers a and b so that x is a square rational number. The family of cubic curves 2 can also be used to establish whether the elliptic curve 3 has or does not have a rational number.

Proof of Beal's conjecture

Relationship between the terms of Beal's conjecture

In a paper entitled simple algebraic proofs of Fermat's last theorem (Buya, 2017a) the author of this paper showed that there exists an algebraic relationship between the terms of Fermat's Diophantine equation. The algebraic equations in the paper can be used to prove Beal conjecture. There exists an algebraic relationship between the terms of Beal's conjecture problem. The proposed proof does not require any Galois representation or use of elliptic curves.

Consider the equation given by:

$$A^x + B^y = C^z \wedge x, y, z > 2 \quad (1)$$

The terms of the Equation above are related in the following way:

$$B^y = \left(\frac{\frac{A^x}{a} - a}{2}\right)^2 = \left(\frac{A^x}{a}\right)^2 \left(\frac{1 - \frac{a^2}{A^x}}{2}\right)^2 \quad (2)$$

$$C^z = \left(\frac{\frac{A^x}{a} + a}{2}\right)^2 = \left(\frac{A^x}{a}\right)^2 \left(\frac{1 + \frac{a^2}{A^x}}{2}\right)^2 \quad (3)$$

To prove the Beal's conjecture, the cases

$$a = \frac{A^x}{m^x} \wedge a, m \in \mathrm{N} \quad (4)$$

will be considered.

Substituting 4 into 2 and 3:

$$B^y = \left(\frac{\frac{A^x}{a} - a}{2}\right)^2 = (m^x)^2 \left(\frac{1 - \frac{A^x}{m^{2x}}}{2}\right)^2 = m^{2x} \left(\frac{m^{2x} - A^x}{2m^{2x}}\right)^2 = \frac{(m^{2x} - A^x)^2}{4m^{2x}} = \frac{(m^{2x} - am^x)^2}{4m^{2x}}$$

$$B = m^{2x/y} \times \left(\frac{1 - \frac{A^x}{m^{2x}}}{2}\right)^{2/y} \quad (5)$$

$$C^z = \left(\frac{\frac{A^x}{a} + a}{2}\right)^2 = (m^x)^2 \left(\frac{1 + \frac{A^x}{m^{2x}}}{2}\right)^2 = (m^x)^2 \left(\frac{m^{2x} + A^x}{2m^{2x}}\right)^2 = \frac{(m^{2x} + am^x)^2}{4m^{2x}} \rightarrow$$

$$C = m^{2x/z} \times \left(\frac{1 + \frac{A^x}{m^{2x}}}{2}\right)^{2/z} \quad (6)$$

From Equation 4:

$$A^x = am^x \rightarrow$$
$$A = a^{1/x} m \quad (7)$$

Substituting 4, 5 and 6 into 1:

$$am^x + \frac{(m^{2x} - am^x)^2}{4m^{2x}} = \frac{(m^{2x} - am^x)^2}{4m^{2x}} \quad (8)$$

For validity of Beal's conjecture if:

$$m^{x/y} \times \left(\frac{1 - \frac{A^x}{m^{2x}}}{2}\right)^{2/y} = u \quad (9)$$

$$m^{x/z} \times \left(\frac{1 + \frac{A^x}{m^{2x}}}{2}\right)^{2/z} = v \quad (10)$$

Substituting equations 9 and 10 into 5 and 6:

$$B = m^{x/y} u \quad (11)$$
$$C = m^{x/z} v \quad (12)$$

The set of equations 7, 11 and 12 suggest that A, B and C at least share a common prime number validating Beal's conjecture.

Example 1

In Equations 7, 11 and 12 if m = 16, x = 3 y = 4 z = 3 then:

$$A^x = 16^3 a$$
$$B^y = 16^3 u^4$$
$$C^z = 16^3 v^4$$

$$16^3 a + 16^3 u^4 = 16^3 v^4 \rightarrow$$
$$16^3 (a + u^4) = 16^3 v^4$$

If we take v = 2, a = 8 and u^4 = 8 we obtain the Equation:

$$32^3 + 32^3 = 16^4$$

The common prime factor in this case is 2.

Example 2

In Equations 7, 11 and 12 if m = 3, x = 6 y = 3 and z = 2 then:

$$A^x = 3^6 a$$
$$B^y = 3^6 u^3$$
$$C^z = 3^8 v^2$$
$$3^6 (a + u^3) = 3^8 v^2$$

If we take v = 1, a = 1 and u = 2 we end up getting the identity:

$$3^6 + 18^3 = 3^8$$

The common prime factor in this example is 3. Thus the above derived relationships show that whenever the powers of A, B and C are greater than 2 then the Beal's conjecture identity will always have a common prime factor shared by A, B and C. Thus, Beal's conjecture is proved.

To completely validate the conjecture other cases will be brought into consideration.
The case x = y = z = 2 will be considered. This is to say we will consider the Pythagorean Diophantine equation. In such a case, the algebraic relationships between A, B and C is given by:

$$B=\frac{\frac{A^2}{a}-a}{2}=\frac{A^2}{a}\left(\frac{1}{2}-\frac{a^2}{A^2}\right)\vdots\wedge a>A\wedge a=\frac{A}{r}\wedge r\in N \quad (13)$$

Equation 13 can be further simplified.

$$B=\frac{\frac{A^2}{a}-a}{2}=Ar\left(\frac{1}{2}-\frac{1}{r^2}\right)=\frac{A}{2r}(r^2-2) \quad (14)$$

$$C=\frac{\frac{A^2}{a}+a}{2}=\frac{A^2}{a}\left(\frac{1}{2}+\frac{a^2}{A^2}\right)=Ar\left(\frac{1}{2}+\frac{1}{r^2}\right)=\frac{A}{2r}(r^2+2) \quad (15)$$

Substituting 14 and 15 into 1:

$$A^2+\left(\frac{\frac{A^2}{a}-a}{2}\right)^2=\left(\frac{\frac{A^2}{a}+a}{2}\right)^2 \quad (16)$$

$$A^2+A^2r^2\left(\frac{1}{2}-\frac{1}{r^2}\right)^2=A^2r^2\left(\frac{1}{2}+\frac{1}{r^2}\right)^2$$

Consider the algebraic fraction $\frac{r^2-2}{2r}$ of equation 14. If it has a form:

$$\frac{r^2-2}{2r}=\frac{\alpha}{A}\vdots\alpha\in N \quad (17)$$

Then on substituting equation 17 into 14:
$B=\alpha$

The number α may or may not be co-primed with A

Consider the algebraic fraction $\frac{r^2+2}{2r}$ of equation 15. If it has a form:

$$\frac{r^2+2}{2r}=\frac{\phi}{A} \quad (18)$$
$$\phi\in N$$

Then on substituting 18 into 15:
$C=\phi$

The number ϕ may or may not be co-primed with A. The Beal's conjecture is therefore not applicable to Pythagorean Diophantine equations.

The case x = 3, y = z = 2 will be considered.

Consider the general Diophantine equation:

$$A^3+B^2=C^2 \quad (19)$$

The relationship between the terms of the above Equation will be considered.

$$B=\frac{\frac{A^{3/2}}{a}-a}{2}=\frac{A^{3/2}}{a}\left(\frac{1}{2}-\frac{a^2}{A^{3/2}}\right) \quad (20)$$

$$C=\frac{\frac{A^{3/2}}{a}+a}{2}=\frac{A^{3/2}}{a}\left(\frac{1}{2}+\frac{a^2}{A^{3/2}}\right)=B \quad (21)$$

If in Equations 21 and 22:

$$\left(\frac{1}{2}-\frac{a^2}{A^{3/2}}\right)=\frac{\beta a}{A^{3/2}} \quad (22)$$

$$\frac{1}{2}+\frac{a^2}{A^{3/2}}=\frac{\chi a}{A^{3/2}} \quad (23)$$

$$\wedge\beta,\chi\in N$$

Then by the Equations 23 and 24:

$B=\beta\wedge C=\chi$

The number $B=\beta$ may or may not be co-primed with A. The number $C=\chi$ may or may not be co-primed with A. Thus Beal's conjecture does not apply to the cases (x,y,z) = (3,2,2) and (x,y,z) = (2,3,2)

The case (x,y,z) = (2,2,3) will be considered.
In this case the relationship between A, B and C is given by the set of equations below:

$$B=\frac{\frac{A^2}{a}-a}{2}=\frac{A^2}{2a}\left(1-\frac{a^2}{A^2}\right) \quad (24)$$

$$C^{3/2}=\frac{\frac{A^2}{a}+a}{2}=\frac{A^2}{2a}\left(1+\frac{a^2}{A^2}\right)\rightarrow \quad (25)$$

$$C=\frac{A^3}{(2a)^{3/2}}\left(1+\frac{a^2}{A^2}\right)^{3/2}$$

If in Equations 24 and 25:

$$1-\frac{a^2}{A^2}=\frac{2a\varepsilon}{A} \quad (26)$$

$$\varepsilon \in N$$

$$(1+\frac{a^2}{A^2})^{3/2}=\frac{(2a)^{3/2}\varphi}{A^3} \quad (27)$$

$$\varphi \in N$$

Then:

$$B=\varepsilon \wedge C=\varphi$$

In the case above C may or may not be co-primed to A. The Beal's conjecture does not apply to the general case (x, y, z) = $(2,2,3)$.

From the above analysis we note that in cases where x, y and z are greater than two, A, B and C share a common prime number. Thus Beal's conjecture is thus verified.

Conclusion

There exists an algebraic relationship connecting the terms of Beal's conjecture problem. The algebraic relationship as stipulated in equations 7, 11 and 12 of section 4 shows that the terms A, B and C share a common prime factor for all x, y and z as positive integers greater than 2. Thus Beal's conjecture is proved.

CONFLICT OF INTERESTS

The author has not declared any conflict of interests.

REFERENCES

Bennet MA, Chen I, Dahmen SR, Yazdani S (2014). Generalized Fermat Equations: A Miscellany. Simon Fraser University. Not cited

Buya SB (2017a). Simple Algebraic proofs of Fermat's last theorem. Advances in Applied Science Research 8(3): 60:64.

Buya SB (2017b). Solution of the Congruent number problem and proof of Birch Swinnerton-Dyer conjecture. Academia.edu.

Elkies ND (2007). The ABC's of Number theory. The Havard College mathematics Review 1(1).

Mauldin RD (1997). A Generalization of Fermat's last theorem: Beal's conjecture and the Prize Problem. Notices of The AMS 44(11):1436-1437.

Puzzles (n.d.). Problems and puzzles. Retrieved September 26th, 2018, from Puzzle 559 - Maudin/Tijman-Zagier conjecture: http:llwww.primepuzzles.net/puzzles/puzz_559.htm Not cited

Waldschmidt M (2004). Open Diophantine problems. Moscow mathematics 4:245-305.

Wikipedia (2018). Retrieved from Beal's conjecture: https://en.m.wikipedia.org/wiki/Beal-conjecture